

FUNCTIONS WITH BOUNDED SPECTRUM

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ABSTRACT. Let $0 < p \leq \infty$, $f(x) \in L_p(\mathbb{R}^n)$, and $\text{supp } Ff$ be bounded, where F is the Fourier transform. We will prove in this paper that the sequence $\|D^\alpha f\|_p^{1/|\alpha|}$, $\alpha \geq 0$, has the same behavior as the sequence $\sup_{\xi \in \text{supp } Ff} |\xi^\alpha|^{1/|\alpha|}$, $\alpha \geq 0$. In other words, if we know all "far points" of $\text{supp } Ff$, we can wholly describe this behavior without any concrete calculation of $\|D^\alpha f\|_p$, $\alpha \geq 0$. A Paley-Wiener-Schwartz theorem for a nonconvex case, which is a consequence of the result, is given.

1. INTRODUCTION

The following result showing a relation between behavior of the sequence of norms of derivatives of a function and the support of its Fourier transform [2] has been proved: Let $1 \leq p \leq \infty$ and $D^m f(x) \in L_p(\mathbb{R}^1)$, $m = 0, 1, \dots$. Then there always exists the limit

$$d_f = \lim_{m \rightarrow \infty} \|D^m f\|_p^{1/m}$$

and moreover

$$d_f = \sup\{|\xi| : \xi \in \text{supp } \tilde{f}\},$$

where $\tilde{f}(\xi) = Ff(\xi)$ is the Fourier transform of the function $f(x)$.

This result is of value in the theory of Sobolev spaces of infinite order, in particular, in studying imbedding theorems for Sobolev spaces of infinite order [3–5].

The question arises as to what happens for the n -dimensional case? In this paper we give a complete answer to this question. It should be noted that here we do not assume any restriction on geometrical properties of $\text{supp } \tilde{f}$ (which is called the spectrum of f).

We will use the following standard notation: $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$; $D = (D_1, \dots, D_n)$; $D_j = \frac{\partial}{\partial x_j}$, $j = 1, \dots, n$; $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$; $\text{sp}(f) = \text{supp } \tilde{f}$.

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And we presuppose that $0^0 = \frac{0}{0} = 1$, $\frac{\lambda}{0} = \infty$ for $\lambda > 0$, $f(x) \in \mathcal{S}'$, and $f(x) \not\equiv 0$.

2. RESULTS

We will show the following

Theorem 1. *Let $0 < p \leq \infty$, $f(x) \in L_p(\mathbb{R}^n)$ and $\text{sp}(f)$ be bounded. Then*

$$(1) \quad \lim_{|\alpha| \rightarrow \infty} (\|D^\alpha f\|_p / \sup_{\text{sp}(f)} |\xi^\alpha|)^{1/|\alpha|} = 1.$$

To prove Theorem 1 we need the following results: Let $0 < p \leq q \leq \infty$ and $K \subset \mathbb{R}^n$ be compact. Denote by M_{Kp} the class of all functions in $\mathcal{S}' \cap L_p(\mathbb{R}^n)$ such that $\text{sp}(f) \subset K$. The following Nikolsky inequality is well known (the ([9], [10, p. 125]): There exists a constant $C(p, q, K)$ such that

$$\|f\|_q \leq C(p, q, K) \|f\|_p$$

for all $f \in M_{Kp}$.

It follows from the Nikolsky inequality that $M_{Kp} \subset M_{K\infty}$, $0 < p \leq \infty$.

Further, let G be a domain in $\mathbb{R}^n$ and $m \in \mathbb{Z}_+$. Denote by $W_{m,2}(G)$ the classical Sobolev space, i.e., the completion $C^m(G)$ with respect to the norm

$$\|f\|_{m,2} = \left(\sum_{|\alpha| \leq m} \|D^\alpha f\|_{L_2(G)}^2 \right)^{1/2}.$$

And $W_{m,2}^0(G)$ is the subspace of all functions $f(x) \in W_{m,2}(G)$ such that the zero extension of $f(x)$ outside G belongs to $W_{m,2}(\mathbb{R}^n)$. For $s \in \mathbb{R}$, we put

$$H_{(s)} = \left\{ f \in \mathcal{S}' : \|f\|_{(s)} = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |Ff(\xi)|^2 d\xi \right)^{1/2} < \infty \right\}.$$

Then

$$H_{(k)} = W_{k,2}(\mathbb{R}^n) \quad (\text{topological imbedding})$$

if $k \in \mathbb{Z}_+$ (see, for example, [1, p. 45; 6, p. 53; 7, 7.9.1]).

Proof of Theorem 1. We divide the proof into three cases.

Case 1 ($1 \leq p < \infty$). We first establish the following inequality

$$(2) \quad \lim_{|\alpha| \rightarrow \infty} (\|D^\alpha f\|_p / |\xi^\alpha|)^{1/|\alpha|} \geq 1$$

for any point $\xi \in \text{sp}(f)$.

Actually, let $\xi^0 \in \text{sp}(f)$, $\xi_j^0 \neq 0$, $j = 1, \dots, n$. (It is easy to show later that there exist such points because of $p < \infty$.) For the sake of convenience, we assume that $\xi_j^0 > 0$, $j = 1, \dots, n$. We fix a number $0 < \epsilon < \frac{1}{2} \min_{1 \leq j \leq n} \xi_j^0$ and choose a domain G with a smooth boundary such that $\xi^0 \in G$ and $G \subset \{\xi : \xi_j^0 - \epsilon \leq \xi_j \leq \xi_j^0 + \epsilon, j = 1, \dots, n\}$. Further we fix a function $\tilde{v}(\xi) \in C_0^\infty(G)$ such that $\xi^0 \in \text{supp}(\tilde{v}\tilde{f})$. Then

$$(3) \quad \langle \tilde{v}(\xi) \tilde{f}(\xi), \tilde{w}(\xi) \rangle = \langle f(x), \varphi(x) \rangle,$$

where $\tilde{w}(\xi) \in C_0^\infty(G)$ is an arbitrary function, $\varphi(x) = \check{v} * \check{w}(x)$, and $\check{u}(x) = u(-x)$. The distribution $\tilde{v}(\xi)\tilde{f}(\xi)$ has a compact support; therefore, it can be represented in the form

$$\tilde{v}(\xi)\tilde{f}(\xi) = \sum_{|\alpha| \leq m} D^\alpha h_\alpha(\xi),$$

where m is a nonnegative integer and $h_\alpha(\xi)$ are ordinary functions in G . Without loss of generality we may assume that $m \geq 2n$.

It is well known that the Dirichlet problem for the elliptic differential equation

$$L_{2m}\tilde{z}(\xi) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha (D^\alpha \tilde{z}(\xi)) = \tilde{v}(\xi)\tilde{f}(\xi)$$

has a (unique) solution $\tilde{z}(\xi) \in W_{m,2}^0(G)$ (see, for example, [6, p. 82]). Since (3), we obtain

$$(4) \quad \langle \tilde{z}(\xi), L_{2m}\tilde{w}(\xi) \rangle = \langle f(x), \varphi(x) \rangle$$

for all $\tilde{w}(\xi) \in C_0^\infty(G)$. The left side of (4) admits a closure up to an arbitrary function $\tilde{w}(\xi) \in W_{m,2}^0(G)$. Hence, replacing $\tilde{w}(\xi)$ by $\xi^\alpha \tilde{w}(\xi)$, we get

$$(5) \quad \langle \tilde{z}(\xi), L_{2m}(\xi^\alpha \tilde{w}(\xi)) \rangle = (-i)^{|\alpha|} \langle D^\alpha f(x), \varphi(x) \rangle$$

for all $\tilde{w}(\xi) \in W_{m,2}^0(G)$.

Now let $\tilde{w}_0(\xi) \in W_{m,2}^0(G)$ be the solution of the equation $L_{2m}\tilde{w}_0(\xi) = \overline{\tilde{z}(\xi)}$. Since $0 \notin G$, we get

$$L_{2m}(\xi^\alpha \tilde{w}_0(\xi)) = \prod_{j=1}^n (\xi_j^0 - 2\epsilon)^{\alpha_j} \overline{\tilde{z}(\xi)},$$

where $\tilde{w}_\alpha(\xi) = \prod_{j=1}^n (\xi_j^0 - 2\epsilon)^{\alpha_j} \xi^{-\alpha} \tilde{w}_0(\xi)$ and $\alpha \geq 0$. Therefore, it follows from (5) that

$$(6) \quad \prod_{j=1}^n (\xi_j^0 - 2\epsilon)^{\alpha_j} \langle \tilde{z}(\xi), \overline{\tilde{z}(\xi)} \rangle \leq \|D^\alpha f\|_p \|v\|_1 \|w_\alpha\|_q,$$

where $1/p + 1/q = 1$.

On the other hand, there exists a constant $C > 0$ such that

$$(7) \quad \|v\|_1 \|w_\alpha\|_q \leq C, \quad \alpha \geq 0.$$

Indeed, let $|\beta| \leq 2n$. Using

$$x^\beta w_\alpha(x) = (-i)^{|\beta|} \prod_{j=1}^n (\xi_j^0 - 2\epsilon)^{\alpha_j} \int_G e^{ix\xi} D^\beta (\xi^{-\alpha} \tilde{w}_0(\xi)) d\xi,$$

the Leibniz formula, and the definition of G , we get

$$\sup_{\mathbb{R}^n} |x^\beta w_\alpha(x)| \leq C_1 \prod_{j=1}^n \left(\frac{\xi_j^0 - 2\epsilon}{\xi_j^0 - \epsilon} \right)^{\alpha_j} \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \prod_{k=1}^n \alpha_k \cdots (\alpha_k + \gamma_k - 1),$$

where

$$C_1 = \max \left\{ \int_G |\xi^{-\gamma} D^{\beta-\gamma} \tilde{w}_0(\xi)| d\xi : \gamma \leq \beta, |\beta| \leq 2n \right\}.$$

On the other hand, since

$$\prod_{k=1}^n \alpha_k \cdots (\alpha_k + \gamma_k - 1) < (|\alpha| + 2n)^{2n}$$

(because of $|\gamma| \leq |\beta| \leq 2n$),

$$2^{|\beta|} = \sum_{\gamma \leq \beta} \binom{\beta}{\gamma}$$

and

$$\lim_{|\alpha| \rightarrow \infty} (|\alpha| + 2n)^{2n} \prod_{j=1}^n \left(\frac{\xi_j^0 - 2\epsilon}{\xi_j^0 - \epsilon} \right)^{\alpha_j} = 0,$$

we obtain

$$\sup_{x \in \mathbb{R}^n} |x^\beta \omega_\alpha(x)| \leq C_2$$

for all $|\beta| \leq 2n$ and $\alpha \geq 0$. Therefore, there is an absolute constant C_3 such that

$$\sup_{\mathbb{R}^n} (1 + x_1^2) \cdots (1 + x_n^2) |w_\alpha(x)| \leq C_3, \quad \alpha \geq 0.$$

So we have proved (7) with $C = C_3 \pi^n \|v\|_1$. Combining (6) and (7) we obtain

$$1 \leq \lim_{|\alpha| \rightarrow \infty} \left(\|D^\alpha f\|_p \prod_{j=1}^n (\xi_j^0 - 2\epsilon)^{-\alpha_j} \right)^{1/|\alpha|}.$$

Therefore, since $\epsilon > 0$ is arbitrarily chosen and

$$\left[\prod_{j=1}^n \left(\frac{\xi_j^0 - 2\epsilon}{\xi_j^0} \right)^{-\alpha_j} \right]^{1/|\alpha|} \leq \max_{1 \leq j \leq n} \frac{\xi_j^0}{\xi_j^0 - 2\epsilon},$$

we obtain (2) (with $\xi = \xi^0$) by letting $\epsilon \rightarrow 0$.

Now we prove (2) for “zero points”: Let $\xi^0 \in \text{sp}(f)$, $\xi^0 \neq 0$, and $\xi_1^0 \cdots \xi_n^0 = 0$. For the sake of convenience, we assume that $\xi_j^0 > 0$, $j = 1, \dots, k$, and $\xi_{k+1}^0 = \cdots = \xi_n^0 = 0$ ($1 \leq k < n$). Then it is enough to show (2) only for indices α such that $\alpha_{k+1} = \cdots = \alpha_n = 0$. Then the proof is analogous to the above one after the following modification of choosing ϵ : We fix a number $0 < \epsilon < \frac{1}{2} \min_{1 \leq j \leq k} \xi_j^0$.

Second we prove that

$$(8) \quad \lim_{|\alpha| \rightarrow \infty} \left(\|D^\alpha f\|_p / \sup_{\text{sp}(f)} |\xi^\alpha| \right)^{1/|\alpha|} \geq 1.$$

Assume the contrary, that there exists a subsequence I_1 such that

$$(9) \quad (I_1) \quad \lim_{|\alpha| \rightarrow \infty} \left(\|D^\alpha f\|_p / \sup_{\text{sp}(f)} |\xi^\alpha| \right)^{1/|\alpha|} < 1,$$

where the symbol (I_1) in (9) means that we take the limit only for $\alpha \in I_1$. Then there exist a subsequence $I_2 \subset I_1$ and numbers $0 \leq \beta_j \leq 1$, $j = 1, \dots, n$ such that $|\beta| = 1$ and

$$(10) \quad (I_2) \quad \lim_{|\alpha| \rightarrow \infty} \frac{\alpha_j}{|\alpha|} = \beta_j, \quad j = 1, \dots, n.$$

Now we prove

$$(11) \quad \lim_{\gamma \rightarrow \beta} \sup_{\text{sp}(f)} |\xi^\gamma| = \sup_{\text{sp}(f)} |\xi^\beta|$$

if $\gamma \in \mathbb{R}_+^n$ and $\gamma \rightarrow \beta$.

Indeed, given $h > 1$, there exists $\epsilon > 0$ such that $h\gamma \geq \beta$ for $\gamma \in \mathbb{R}_+^n$ and $|\gamma - \beta| \leq \epsilon$. Further, let $|\xi| \leq M$ for all $\xi \in \text{sp}(f)$. Then for $\xi \in \text{sp}(f)$ and $\gamma \in \mathbb{R}_+^n$, $|\gamma - \beta| \leq \epsilon$ we obtain

$$|\xi^\gamma| = |\xi^{\gamma-\beta/h}| |\xi^\beta|^{1/h} \leq M^{|\gamma-\beta/h|} \sup_{\text{sp}(f)} |\xi^\beta|^{1/h}.$$

Therefore

$$\overline{\lim}_{\gamma \rightarrow \beta} \sup_{\text{sp}(f)} |\xi^\gamma| \leq M^{|\beta|(1-1/h)} \sup_{\text{sp}(f)} |\xi^\beta|^{1/h}.$$

Letting $h \rightarrow 1$, we get

$$\overline{\lim}_{\gamma \rightarrow \beta} \sup_{\text{sp}(f)} |\xi^\gamma| \leq \sup_{\text{sp}(f)} |\xi^\beta|.$$

To prove (11) it remains to show that

$$(12) \quad \underline{\lim}_{\gamma \rightarrow \beta} \sup_{\text{sp}(f)} |\xi^\gamma| \geq \sup_{\text{sp}(f)} |\xi^\beta|.$$

Let $\xi^* \in \text{sp}(f)$ such that $|\xi^{*\beta}| = \sup_{\text{sp}(f)} |\xi^\beta|$. Then it follows from $p < \infty$ that the

distribution $\tilde{f}(\xi)$ cannot concentrate on the hyperplanes $\xi_j = 0$, $j = 1, \dots, n$ (this fact will be shown later). Therefore, $|\xi^{*\beta}| > 0$. Furthermore, let η be an arbitrary point of $\text{sp}(f)$. Then we will show later that the restriction of the distribution $\tilde{f}(\xi)$ on any neighborhood of η also does not concentrate on the hyperplanes $\xi_j = 0$, $j = 1, \dots, n$. Therefore, there exists a sequence $m\xi \in \text{sp}(f)$, $m \geq 1$, such that $m\xi_j \neq 0$, $j = 1, \dots, n$, for any $m \geq 1$ and $m\xi \rightarrow \xi^*$, $m \rightarrow \infty$. Then

$$\sup_{\text{sp}(f)} |\xi^\gamma| \geq |m\xi^\gamma|$$

for any $m \geq 1$. Hence,

$$\underline{\lim}_{\gamma \rightarrow \beta} \sup_{\text{sp}(f)} |\xi^\gamma| \geq \underline{\lim}_{\gamma \rightarrow \beta} |m\xi^\gamma| = |m\xi^\beta|.$$

Letting $m \rightarrow \infty$, we obtain (12) and then (11).

Further, given $\lambda > 1$, there is a number $k \geq 1$ such that $\lambda |k\xi^\beta| \geq |\xi^{*\beta}|$. Therefore, it follows from (10)–(11) and (2) that

$$\begin{aligned} (I_2) \quad \underline{\lim}_{|\alpha| \rightarrow \infty} (\|D^\alpha f\|_p / \sup_{\text{sp}(f)} |\xi^\alpha|)^{1/|\alpha|} &= (I_2) \quad \underline{\lim}_{|\alpha| \rightarrow \infty} \|D^\alpha f\|_p^{1/|\alpha|} / |\xi^{*\beta}| \\ &\geq (I_2) \quad \frac{1}{\lambda} \underline{\lim}_{|\alpha| \rightarrow \infty} \|D^\alpha f\|_p^{1/|\alpha|} / |k\xi^\beta| \\ &= (I_2) \quad \frac{1}{\lambda} \underline{\lim}_{|\alpha| \rightarrow \infty} \|D^\alpha f\|_p / |k\xi^\beta|^{1/|\alpha|} \geq \frac{1}{\lambda}. \end{aligned}$$

This contradicts (9) by letting $\lambda \rightarrow 1$. Thus we have proved (8).

Finally, we will show

$$(13) \quad \lim_{|\alpha| \rightarrow \infty} (\|D^\alpha f\|_p / \sup_{\text{sp}(f)} |\xi^\alpha|)^{1/|\alpha|} \leq 1.$$

We fix a domain $G \supset \text{sp}(f)$ and a function $\psi \in C_0^\infty(G)$ such that $\psi(\xi)$ equals 1 in some neighborhood of $\text{sp}(f)$. Further, let $0 < q \leq 1$. We put $h_\alpha(\xi) = \psi(\xi)\xi^\alpha$, $\alpha \geq 0$. Then it follows from Hölder's inequality that for any $s > n(1/q - 1/2)$

$$\begin{aligned} \|F^{-1}h_\alpha\|_q^q &= \int (|\tilde{h}_\alpha(\xi)|^2)^{q/2} d\xi \\ &\leq \left(\int |\tilde{h}_\alpha(\xi)|^2 (1 + |\xi|^2)^s d\xi \right)^{q/2} \times \left(\int (1 + |\xi|^2)^{-sq/(2-q)} d\xi \right)^{1-q/2}. \end{aligned}$$

Therefore,

$$(14) \quad \|F^{-1}h_\alpha\|_q \leq C' \|h_\alpha\|_{(s)},$$

where $C' = C'(s, q)$ is independent of h_α .

Combining (14), the topological equality $H_{(k)} = W_{k,2}(\mathbb{R}^n)$, and

$$\|D^\alpha f\|_p = \|F^{-1}(\psi(\xi)\xi^\alpha) * f\|_p \leq \|F^{-1}(\psi(\xi)\xi^\alpha)\|_1 \|f\|_p,$$

we get

$$(15) \quad \|D^\alpha f\|_p \leq C \|\psi(\xi)\xi^\alpha\|_{k,2} \|f\|_p, \quad \alpha \geq 0,$$

where C is independent of f and α and $k = [\frac{n}{2}] + 1$.

Given the Leibniz formula we get a constant $\bar{C}_1 = C_1(\psi, k)$ such that

$$(16) \quad \|\psi(\xi)\xi^\alpha\|_{k,2} \leq C_1 |\alpha|^k \sup\{\sup_G |\xi^{\alpha-\gamma}| : \gamma \leq \alpha, |\gamma| \leq k\}, \quad \alpha \geq 0.$$

On the other hand, by an argument analogous to the previous one, we get

$$(17) \quad \lim_{|\alpha| \rightarrow \infty} (\sup\{\sup_G |\xi^{\alpha-\gamma}| : \gamma \leq \alpha, |\gamma| \leq k\})^{1/|\alpha|} / \sup_G |\xi^\alpha|^{1/|\alpha|} = 1.$$

Actually, assume the contrary, that there exist a subsequence I_1 and a number $\delta > 1$ such that

$$(18) \quad \sup\{\sup_G |\xi^{\alpha-\gamma}|^{1/|\alpha|} : \gamma \leq \alpha, |\gamma| \leq k\} \geq \delta \sup_G |\xi^\alpha|^{1/|\alpha|}, \quad \alpha \in I_1.$$

Therefore, there are a subsequence $I_2 \subset I_1$, numbers $0 \leq \beta_j \leq 1$, $j = 1, \dots, n$, and an index γ^0 , $|\gamma^0| \leq k$, such that $|\beta| = 1$ and

$$(I_2) \quad \lim_{|\alpha| \rightarrow \infty} \frac{\alpha_j - \gamma_j^0}{|\alpha|} = \beta_j, \quad j = 1, \dots, n,$$

and

$$\sup \left\{ \sup_G |\xi^{\alpha-\gamma}|^{1/|\alpha|} : \gamma \leq \alpha, |\gamma| \leq k \right\} = \sup_G |\xi^{\alpha-\gamma^0}|^{1/|\alpha|}$$

for all $\alpha \in I_2$. Therefore, by an argument analogous to that used for the proof of (11), we get

$$(I_2) \quad \lim_{|\alpha| \rightarrow \infty} \sup_G |\xi^{\alpha-\gamma^0}|^{1/|\alpha|} = (I_2) \quad \lim_{|\alpha| \rightarrow \infty} \sup_G |\xi^\alpha|^{1/|\alpha|} = \sup_G |\xi^\beta| > 0,$$

which contradicts (18). Thus we have proved (17).

Combining (15)–(17) we obtain

$$(19) \quad \overline{\lim}_{|\alpha| \rightarrow \infty} \|D^\alpha f\|_p^{1/|\alpha|} / \sup_G |\xi^\alpha|^{1/|\alpha|} \leq 1.$$

Now we assume the contrary that (13) does not hold. Then there exist a subsequence J and numbers $\lambda > 1$, $0 \leq \beta_j \leq 1$, $j = 1, \dots, n$, such that $|\beta| = 1$ and

$$(J) \quad \lim_{|\alpha| \rightarrow \infty} \|D^\alpha f\|_p^{1/|\alpha|} / \sup_{\text{sp}(f)} |\xi^\alpha|^{1/|\alpha|} = \lambda,$$

$$(J) \quad \lim_{|\alpha| \rightarrow \infty} \frac{\alpha_j}{|\alpha|} = \beta_j, \quad j = 1, \dots, n.$$

Therefore, given the validity of (11) when $\text{sp}(f)$ is replaced by G (which can be proved analogously because G is open) and (19) we get

$$\sup_G |\xi^\beta| / \sup_{\text{sp}(f)} |\xi^\beta| \geq \lambda$$

for any domain $G \supset \text{sp}(f)$, which is impossible because of $\sup_{\text{sp}(f)} |\xi^\beta| > 0$. The proof of Case 1 is complete.

Case 2 ($p = \infty$). This is the most complicated case. It should be noted that many facts used in the proof of Case 1 are false in this case (for example, equality (11)). We first prove that if $\sup_{\text{sp}(f)} |\xi^\alpha| = 0$, then $D^\alpha f(x) \equiv 0$ (for the same α). Indeed, without loss of generality we may assume that $\alpha_j \neq 0$, $j = 1, \dots, k$, and $\alpha_{k+1} = \dots = \alpha_n = 0$ ($1 \leq k \leq n$). Then the distribution $\tilde{f}(\xi)$ concentrates on the hyperplanes $\xi_j = 0$, $j \in \{1, \dots, k\} = I$.

For each $j \in I$ we put

$$G_j = \{\xi \in \mathbb{R}^n : \xi_i \neq 0, i \in I \setminus \{j\}\}.$$

Then G_j is open. And let $\tilde{f}_1(\xi)$ be the restriction of $\tilde{f}(\xi)$ on $\bigcup_{j \in I} G_j$. Then using a partition of unity (see, for example, [7, Theorem 1.4.5]), we get

$$\tilde{f}_1(\xi) = \sum_{j=1}^k \varphi_j(\xi) \tilde{f}(\xi),$$

where $\varphi_j(\xi) \in C_0^\infty(G_j)$, $j \in I$.

The distribution $\varphi_j(\xi) \tilde{f}(\xi)$ concentrates on the hyperplanes $\xi_j = 0$. Therefore, taking account of a remark on Theorem 2.3.5 mentioned in Example 5.1.2 in [7], we get

$$(20) \quad F^{-1}(\varphi_j \tilde{f})(x) = \sum_{\ell=0}^N g_\ell(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \cdot (-ix_j)^\ell,$$

where N is the order of the distribution $\tilde{f}(\xi)$ ($N < \infty$ because $\text{supp } \tilde{f}$ is compact), $g_\ell(\xi_1, \dots, \xi_{j-1}, \xi_{j+1}, \dots, \xi_n)$, $\ell = 0, 1, \dots, N$ —distributions with compact support.

On the other hand, we have

$$\|F^{-1}(\varphi_j \tilde{f})\|_\infty = \|F^{-1} \varphi_j * f\|_\infty \leq \|F^{-1} \varphi_j\|_1 \|f\|_\infty < \infty.$$

Therefore, (20) is possible only if $\ell = 0$. Consequently, the function $F^{-1}(\varphi_j \tilde{f})(x)$ is independent of x_j . Hence, since $\alpha_j \neq 0$, we get

$$D^\alpha F^{-1}(\varphi_j \tilde{f})(x) \equiv 0, \quad j \in I,$$

and then $D^\alpha f_1(x) \equiv 0$. Therefore, to prove $D^\alpha f(x) \equiv 0$, it is enough to show that $D^\alpha(f - f_1)(x) \equiv 0$.

Further, let $\xi \in \mathbb{R}^n \setminus \bigcup_{j \in I} G_j$. Then there exist at least two indices $i, j \in I$

such that $\xi_i = \xi_j = 0$. Therefore, the distribution $\tilde{f}(\xi) - \tilde{f}_1(\xi)$ will concentrate on the hyperplanes $\xi_i = \xi_j = 0$, $i, j \in I$, $i \neq j$.

Let $i, j \in I$, $i \neq j$. We put

$$G_{ij} = \{\xi \in \mathbb{R}^n : \xi_\ell \neq 0, \ell \in I \setminus \{i, j\}\}.$$

Then G_{ij} is open. And let $\tilde{f}_2(\xi)$ be the restriction of $\tilde{f}(\xi) - \tilde{f}_1(\xi)$ on the union of the sets G_{ij} , $i, j \in I$, $i \neq j$. Then using the partition of unity, we get

$$\tilde{f}_2(\xi) = \sum_{i, j \in I, i \neq j} \varphi_{ij}(\xi)(\tilde{f}(\xi) - \tilde{f}_1(\xi)),$$

where $\varphi_{ij}(\xi) \in C_0^\infty(G_{ij})$.

The distribution $\varphi_{ij}(\xi)(\tilde{f}(\xi) - \tilde{f}_1(\xi))$ concentrates on the hyperplane $\xi_i = \xi_j = 0$. Therefore, $D^\alpha F^{-1}[\varphi_{ij}(\tilde{f} - \tilde{f}_1)](x) \equiv 0$ because, as shown above, $F^{-1}(\varphi_{ij}(\tilde{f} - \tilde{f}_1))(x)$ is independent of variables x_i, x_j . Consequently, $D^\alpha f_2(x) \equiv 0$. Therefore, to prove $D^\alpha f(x) \equiv 0$, it is enough to show that $D^\alpha(f - f_1 - f_2)(x) \equiv 0$.

Now let $\xi \in \mathbb{R}^n \setminus \bigcup\{G_{ij} : i, j \in I, i \neq j\}$. Then there are at least three indices $i_1, i_2, i_3 \in I$ such that $\xi_{i_1} = \xi_{i_2} = \xi_{i_3} = 0$. Therefore, the distribution $\tilde{f}(\xi) - \tilde{f}_1(\xi) - \tilde{f}_2(\xi)$ concentrates on the hyperplanes $\xi_{i_1} = \xi_{i_2} = \xi_{i_3} = 0$, $i_1, i_2, i_3 \in I$, $i_1 \neq i_2 \neq i_3$.

Again for $i_1, i_2, i_3 \in I$, $i_1 \neq i_2 \neq i_3$, we put

$$G_{i_1 i_2 i_3} = \{\xi \in \mathbb{R}^n : \xi_j \neq 0, j \in I \setminus \{i_1, i_2, i_3\}\}$$

and call $\tilde{f}_3(\xi)$ the restriction of $\tilde{f}(\xi) - \tilde{f}_1(\xi) - \tilde{f}_2(\xi)$ on the union of the sets $G_{i_1 i_2 i_3}$, $i_1, i_2, i_3 \in I$, $i_1 \neq i_2 \neq i_3$. Then we have again $D^\alpha f_3(x) \equiv 0$. So it is enough to prove $D^\alpha(f - f_1 - f_2 - f_3)(x) \equiv 0$, where $\tilde{f}(\xi) - \tilde{f}_1(\xi) - \tilde{f}_2(\xi) - \tilde{f}_3(\xi)$ concentrates on the hyperplanes $\xi_{i_1} = \xi_{i_2} = \xi_{i_3} = \xi_{i_4} = 0$, $i_1, i_2, i_3, i_4 \in I$, $i_1 \neq i_2 \neq i_3 \neq i_4$. Repeating the above arguments, we obtain the distribution $\tilde{f}(\xi) - \tilde{f}_1(\xi) - \dots - \tilde{f}_{k-1}(\xi)$ which concentrates on the hyperplane $\xi_1 = \dots = \xi_k = 0$ and $D^\alpha f(x) \equiv 0$ if $D^\alpha(f - f_1 - \dots - f_{k-1})(x) \equiv 0$. The last fact is clear because, as shown above, $(f - f_1 - \dots - f_{k-1})(x)$ does not depend on variables $x_1, \dots, x_k$. Thus we have proved $D^\alpha f(x) \equiv 0$.

By the result just obtained, it is enough to show (1) only for multi-indices $\alpha \geq 0$ such that $\sup_{\text{sp}(f)} |\xi^\alpha| > 0$ and denote by P the set of all such as multi-indices.

Second we notice that inequalities (2) and (19) have been proved for $1 \leq p \leq \infty$.

Next we prove that

$$(21) \quad (P) \lim_{|\alpha| \rightarrow \infty} (\|D^\alpha f\|_\infty / \sup_{\text{sp}(f)} |\xi^\alpha|)^{1/|\alpha|} \geq 1.$$

Assume to the contrary, that there exist a subsequence $I \subset P$, a number $\lambda < 1$, and a vector $\beta \geq 0$, $|\beta| = 1$ such that

$$(22) \quad (I) \lim_{|\alpha| \rightarrow \infty} (||D^\alpha f||_\infty / \sup_{\text{sp}(f)} |\xi^\alpha|)^{1/|\alpha|} < \lambda,$$

$$(23) \quad (I) \lim_{|\alpha| \rightarrow \infty} \frac{\alpha}{|\alpha|} = \beta.$$

Note that

$$(24) \quad (I) \lim_{|\alpha| \rightarrow \infty} \sup_{\text{sp}(f)} |\xi^\alpha|^{1/|\alpha|} > 0.$$

Indeed, assume to the contrary, that there exists a subsequence $J \subset I$ such that

$$(25) \quad (J) \lim_{|\alpha| \rightarrow \infty} \sup_{\text{sp}(f)} |\xi^\alpha|^{1/|\alpha|} = 0.$$

For any $1 \leq k \leq n$ and $i_1, \dots, i_k \in \{1, \dots, n\}$ we put

$$T_{i_1 \dots i_k} = \{\alpha \geq 0 : \alpha_{i_1} \neq 0, \dots, \alpha_{i_k} \neq 0 \text{ and } \alpha_j = 0 \text{ if } j \notin \{i_1, \dots, i_k\}\}.$$

Then there exist $1 \leq k \leq n$ and $i_1, \dots, i_k \in \{1, \dots, n\}$ such that $J_{i_1 \dots i_k} = J \cap T_{i_1 \dots i_k}$ is unbounded. Therefore, clearly, we get

$$(J_{i_1 \dots i_k}) \lim_{|\alpha| \rightarrow \infty} \sup_{\text{sp}(f)} |\xi^\alpha|^{1/|\alpha|} \geq (J_{i_1 \dots i_k}) \lim_{|\alpha| \rightarrow \infty} |\eta^\alpha|^{1/|\alpha|} > 0,$$

where η is any point of $\text{sp}(f)$ such that $\eta_{i_1} \neq 0, \dots, \eta_{i_k} \neq 0$. This contradicts (25). So we have proved (24).

Further, let ${}_\alpha \xi \in \text{sp}(f) : |{}_\alpha \xi^\alpha| = \sup_{\text{sp}(f)} |\xi^\alpha|$. Then ${}_\alpha \xi_{i_1} \neq 0, \dots, {}_\alpha \xi_{i_k} \neq 0$ for any $\alpha \in J_{i_1 \dots i_k}$ and, by taking a subsequence, without loss of generality we may assume that for some $\xi^* \in \text{sp}(f)$

$$(26) \quad (J_{i_1 \dots i_k}) \lim_{|\alpha| \rightarrow \infty} {}_\alpha \xi = \xi^*.$$

Now we consider two cases of ξ^* :

If $\xi_{i_j}^* \neq 0$, $j = 1, \dots, k$, then, obviously,

$$(J_{i_1 \dots i_k}) \lim_{|\alpha| \rightarrow \infty} |{}_\alpha \xi^\alpha|^{1/|\alpha|} = |\xi^{*\beta}| = (J_{i_1 \dots i_k}) \lim_{|\alpha| \rightarrow \infty} |\xi^{*\alpha}|^{1/|\alpha|}$$

which together with $\xi^* \in \text{sp}(f)$, (2), and (22) implies

$$\begin{aligned} 1 &\leq (J_{i_1 \dots i_k}) \lim_{|\alpha| \rightarrow \infty} (||D^\alpha f||_\infty / |\xi^{*\alpha}|)^{1/|\alpha|} \\ &= (J_{i_1 \dots i_k}) \lim_{|\alpha| \rightarrow \infty} (||D^\alpha f||_\infty / \sup_{\text{sp}(f)} |\xi^\alpha|)^{1/|\alpha|} < \lambda < 1, \end{aligned}$$

which is impossible.

Otherwise, without loss of generality we may assume that $\xi_{i_1}^* = \dots = \xi_{i_m}^* = 0$ and $\xi_{i_{m+1}}^* \neq 0, \dots, \xi_{i_k}^* \neq 0$ for some $1 \leq m \leq k$.

Since (24) and (26), it follows that $\xi^* \neq 0$. Therefore, $m < k$. Further, by virtue of (23)–(24), (26), the definition of ${}_\alpha \xi$, and $\xi_{i_1}^* = \dots = \xi_{i_m}^* = 0$ we obtain $\beta_{i_1} = \dots = \beta_{i_m} = 0$. Since, clearly,

$$\begin{aligned} (J_{i_1 \dots i_k}) \lim_{|\alpha| \rightarrow \infty} |{}_\alpha \xi_{i_{m+1}}^{\alpha_{i_{m+1}}} \dots {}_\alpha \xi_{i_k}^{\alpha_{i_k}}|^{1/|\alpha|} &= |\xi_{i_{m+1}}^{*\beta_{i_{m+1}}} \dots \xi_{i_k}^{*\beta_{i_k}}| \\ &= (J_{i_1 \dots i_k}) \lim_{|\alpha| \rightarrow \infty} |\xi_{i_{m+1}}^{*\alpha_{i_{m+1}}} \dots \xi_{i_k}^{*\alpha_{i_k}}|^{1/|\alpha|}, \end{aligned}$$

there exist $\nu \in J_{i_1 \dots i_k}$ and $N > 0$ such that

$$(27) \quad |\alpha \xi_{i_\ell}| \leq \lambda^{-1} |\nu \xi_{i_\ell}|, \quad \ell = m+1, \dots, k,$$

for all $|\alpha| \geq N$, $\alpha \in J_{i_1 \dots i_k}$

On the other hand, it follows from $\nu \xi_{i_1} \neq 0, \dots, \nu \xi_{i_k} \neq 0$ and

$$(J_{i_1 \dots i_k}) \lim_{|\alpha| \rightarrow \infty} \alpha \xi_{i_j} = \xi_{i_j}^* = 0, \quad j = 1, \dots, m,$$

that there exists $M > 0$ such that

$$|\alpha \xi_{i_j}| \leq |\nu \xi_{i_j}|, \quad j = 1, \dots, m,$$

for all $|\alpha| \geq M$, $\alpha \in J_{i_1 \dots i_k}$. This together with (27) implies

$$|\alpha \xi_{i_j}| \leq \lambda^{-1} |\nu \xi_{i_j}|, \quad j = 1, \dots, k,$$

for all $|\alpha| \geq \max\{M, N\}$, $\alpha \in J_{i_1 \dots i_k}$. Therefore,

$$\sup_{\text{sp}(f)} |\xi^\alpha|^{1/|\alpha|} = |\alpha \xi^\alpha|^{1/|\alpha|} \leq \lambda^{-1} |\nu \xi^\alpha|^{1/|\alpha|}$$

which together with (2) and (22) implies

$$\begin{aligned} 1 &\leq (J_{i_1 \dots i_k}) \lim_{|\alpha| \rightarrow \infty} (||D^\alpha f||_\infty / |\nu \xi^\alpha|)^{1/|\alpha|} \\ &\leq (J_{i_1 \dots i_k}) \lambda^{-1} \lim_{|\alpha| \rightarrow \infty} (||D^\alpha f||_\infty / \sup_{\text{sp}(f)} |\xi^\alpha|)^{1/|\alpha|} < 1. \end{aligned}$$

We thus arrive at a contradiction. So we have proved (21).

Finally, to complete the proof it remains to show that

$$(28) \quad (P) \overline{\lim}_{|\alpha| \rightarrow \infty} (||D^\alpha f||_\infty / \sup_{\text{sp}(f)} |\xi^\alpha|)^{1/|\alpha|} \leq 1.$$

Assume to the contrary, that there exist a subsequence $I \subset P$, a number $h > 1$, and a vector $\beta \geq 0$, $|\beta| = 1$ such that

$$(29) \quad (I) \lim_{|\alpha| \rightarrow \infty} (||D^\alpha f||_\infty / \sup_{\text{sp}(f)} |\xi^\alpha|)^{1/|\alpha|} > h,$$

$$(30) \quad (I) \lim_{|\alpha| \rightarrow \infty} \frac{\alpha}{|\alpha|} = \beta.$$

Notation being as above, we have $1 \leq k \leq n$ and $i_1, \dots, i_k \in \{1, \dots, n\}$ such that $I_{i_1 \dots i_k} = I \cap T_{i_1 \dots i_k}$ is unbounded.

We put

$$\begin{aligned} Q &= \{\eta \in \mathbb{R}^n : \exists \{m\xi\} \subset \text{sp}(f), m\xi_j \neq 0, \\ &\quad j \in \{i_1, \dots, i_k\}, m \geq 1, \lim_{m \rightarrow \infty} m\xi = \eta\}, \\ Q_\delta &= \{x + y : x \in Q, |y| < \delta\}, \quad \delta > 0, \end{aligned}$$

and $H = \mathbb{R}^n \setminus Q$. Then Q is close, H and Q_δ are open.

Therefore, since $\text{sp}(f) \subset Q_\delta \cup H (= \mathbb{R}^n)$, we obtain

$$\tilde{f}(\xi) = \varphi_\delta(\xi) \tilde{f}(\xi) + \psi(\xi) \tilde{f}(\xi), \quad \varphi_\delta \in C_0^\infty(Q_\delta), \psi \in C_0^\infty(H).$$

By an argument analogous to the previous one, we can prove that $D^\alpha F^{-1}(\psi \tilde{f})(x) \equiv 0$ for all $\alpha \in I_{i_1 \dots i_k}$. Hence, it follows from (29) that

$$(31) \quad (I_{i_1 \dots i_k}) \lim_{|\alpha| \rightarrow \infty} (||D^\alpha F^{-1}(\varphi_\delta \tilde{f})||_\infty / \sup_{\text{sp}(f)} |\xi^\alpha|)^{1/|\alpha|} > h$$

for any $\delta > 0$.

On the other hand, by an argument used in the proof of equality (11), we get

$$(32) \quad (I_{i_1 \dots i_k}) \lim_{|\alpha| \rightarrow \infty} \sup_{Q_\delta} |\xi^\alpha|^{1/|\alpha|} = \sup_{Q_\delta} |\xi^\beta|.$$

Further, let $m\theta \in Q_{1/m} : |m\theta^\beta| = \sup_{Q_{1/m}} |\xi^\beta|$, $m \geq 1$. Then there exist a subsequence $\{m_k\}$ (for simplicity of notation we assume that $m_k = k$, $k \geq 1$) and a point $\theta^* \in Q$ such that $m\theta \rightarrow \theta^*$, $m \rightarrow \infty$. Then

$$0 < \sup_Q |\xi^\beta| \leq \lim_{m \rightarrow \infty} |m\theta^\beta| = |\theta^{*\beta}|.$$

Arguing as in the proof of (12) and taking account of $\theta^* \in Q$ and (30) we obtain

$$(33) \quad |\theta^{*\beta}| \leq (I_{i_1 \dots i_k}) \lim_{|\alpha| \rightarrow \infty} \sup_Q |\xi^\alpha|^{1/|\alpha|}.$$

Further, because inequality (19) was proved for $1 \leq p \leq \infty$, we have

$$(34) \quad (I_{i_1 \dots i_k}) \overline{\lim}_{|\alpha| \rightarrow \infty} (\|D^\alpha F^{-1}(\varphi_{1/m}\tilde{f})\|_\infty / \sup_{Q_{1/m}} |\xi^\alpha|)^{1/|\alpha|} \leq 1$$

for any $m \geq 1$.

Now we fix a number $m \geq 1$ such that $|m\theta^\beta| \leq h|\theta^{*\beta}|$. Then combining (31)–(34), we obtain

$$\begin{aligned} 1 &\geq (I_{i_1 \dots i_k}) \overline{\lim}_{|\alpha| \rightarrow \infty} (\|D^\alpha F^{-1}(\varphi_{1/m}\tilde{f})\|_\infty / \sup_{Q_{1/m}} |\xi^\alpha|)^{1/|\alpha|} \\ &= (I_{i_1 \dots i_k}) \overline{\lim}_{|\alpha| \rightarrow \infty} \|\tilde{f}\|_\infty^{1/|\alpha|} / |m\theta^\beta| \\ &\geq (I_{i_1 \dots i_k}) \overline{\lim}_{|\alpha| \rightarrow \infty} h^{-1} \|\tilde{f}\|_\infty^{1/|\alpha|} / |\theta^{*\beta}| \\ &\geq (I_{i_1 \dots i_k}) \overline{\lim}_{|\alpha| \rightarrow \infty} h^{-1} (\|D^\alpha F^{-1}(\varphi_{1/m}\tilde{f})\|_\infty / \sup_Q |\xi^\alpha|)^{1/|\alpha|} \\ &= (I_{i_1 \dots i_k}) \lim_{|\alpha| \rightarrow \infty} h^{-1} (\|D^\alpha F^{-1}(\varphi_{1/m}\tilde{f})\|_\infty / \sup_{\text{sp}(\tilde{f})} |\xi^\alpha|)^{1/|\alpha|} > 1, \end{aligned}$$

which is impossible.

The proof of Case 2 is complete.

Let us now return to prove the fact mentioned in the proof of Case 1 that the distribution $\tilde{f}(\xi)$ does not concentrate on the hyperplanes $\xi_j = 0$, $j = 1, \dots, n$ if $1 \leq p < \infty$. Actually, assume to the contrary, that $\tilde{f}$ concentrates on the hyperplanes $\xi_j = 0$, $j = 1, \dots, n$. Then, notation being as above, we have

$$\tilde{f}_1(\xi) = \sum_{j=1}^n \varphi_j(\xi) \tilde{f}(\xi), \quad \varphi_j \in C_0^\infty(G_j),$$

where $\tilde{f}_1$ is the restriction of $\tilde{f}$ on $\bigcup_{j=1}^n G_j$.

On the other hand, it follows from the Nikolsky inequality that $f(x) \in L_\infty$. Therefore, as shown above, $F^{-1}(\varphi_j \tilde{f})(x)$ is independent of x_j , which is possible only if $F^{-1}(\varphi_j \tilde{f})(x) \equiv 0$ because of

$$\|F^{-1}(\varphi_j \tilde{f})\|_p \leq \|F^{-1}\varphi_j\|_1 \|f\|_p < \infty$$

and $p < \infty$. Therefore, $\tilde{f}$ must concentrate on the hyperplanes $\xi_i = \xi_j = 0$, $i, j \in \{1, \dots, n\}$, $i \neq j$. Repeating the above arguments, taking account of $p < \infty$, we obtain $\text{supp } \tilde{f} \subset \{0\}$, which is impossible because of $p < \infty$ and $f(x) \not\equiv 0$.

Further, let η be an arbitrary point of $\text{sp}(f)$. Then analogously we can prove that the restriction of $\tilde{f}(\xi)$ on any neighborhood of η also cannot concentrate on the hyperplanes $\xi_j = 0$, $j = 1, \dots, n$.

Case 3 ($0 < p < 1$). The inequality

$$\lim_{|\alpha| \rightarrow \infty} (\|D^\alpha f\|_p / \sup_{\text{sp}(f)} |\xi^\alpha|)^{1/|\alpha|} \geq 1$$

follows from the Nikolsky inequality and Case 1. The inverse inequality

$$\overline{\lim}_{|\alpha| \rightarrow \infty} (\|D^\alpha f\|_p / \sup_{\text{sp}(f)} |\xi^\alpha|)^{1/|\alpha|} \leq 1$$

can be proved in the same way as shown above with the following modification of (15):

$$(35) \quad \|D^\alpha f\|_p \leq C \|\psi(\xi) \xi^\alpha\|_{k,2} \|f\|_p, \quad \alpha \geq 0,$$

where $k = [n(\frac{1}{p} - \frac{1}{2})] + 1$.

Let us now prove (35). Given (14), we get

$$(36) \quad \|F^{-1} h_\alpha\|_p \leq C' \|h_\alpha\|_{(k)}, \quad \alpha \geq 0,$$

where C' is independent of f and α and notation is as above.

On the other hand, given

$$\begin{aligned} \text{supp } F(F^{-1} h_\alpha(\cdot) f(x - \cdot)) &\subset \text{supp } h_\alpha + \text{supp } F(f(x - \cdot)) \\ &= \text{supp } h_\alpha - \text{supp } Ff \subset G - \text{sp}(f), \end{aligned}$$

the Nikolsky inequality, and (36), we get $F^{-1} h_\alpha \in L_\infty$, $F^{-1} h_\alpha(\cdot) f(x - \cdot) \in L_p$ for any $x \in \mathbb{R}^n$, $\alpha \geq 0$, and

$$\begin{aligned} |(F^{-1} h_\alpha F f)(x)|^p &\leq \left(\int |F^{-1} h_\alpha(y) f(x - y)| dy \right)^p \\ &\leq C_1^p \int |F^{-1} h_\alpha(y)|^p |f(x - y)|^p dy, \end{aligned}$$

where $C_1 = C_1(p, G - \text{sp}(f))$. Therefore, given (36) we obtain

$$\begin{aligned} \|D^\alpha f\|_p &= \|F^{-1} h_\alpha F f\|_p \leq C_1 \|F^{-1} h_\alpha\|_p \|f\|_p \\ &\leq C_2 \|h_\alpha\|_{(k)} \|f\|_p \leq C \|h_\alpha\|_{k,2} \|f\|_p \end{aligned}$$

for all $\alpha \geq 0$. So we have proved (35).

The proof of Theorem 1 is complete.

Remark 1. By an easier way we can prove Theorem 1 for functions defined on torus $\mathbb{T}^n$.

Remark 2. Theorem 1 still holds for the case of fractional derivatives. And it can be extended to the cases of other derivatives as Riesz' or Bessel's ones (see [8]).

Remark 3. Equality (1) is not true if $\text{sp}(f)$ is unbounded. However, we have

Theorem 2. Let $0 < p \leq \infty$, $f(x) \in L_p(\mathbb{R}^n)$, and $\text{sp}(f)$ with respect to $\xi_1, \dots, \xi_k$ ($1 \leq k \leq n$) be bounded. Then $D^\nu f(x) \in L_p(\mathbb{R}^n)$ for all $\nu = (\nu_1, \dots, \nu_k, 0, \dots, 0) \in \mathbb{Z}_+^n$ and

$$\lim_{|\nu| \rightarrow \infty} (||D^\nu f||_p / \sup_{\text{sp}(f)} |\xi^\nu|)^{1/|\nu|} = 1.$$

3. AN APPLICATION

Now let us apply Theorem 1 to obtain a nonconvex case of the Paley–Wiener–Schwartz theorem. For this purpose we have to introduce a notion on a set generated by a number sequence: Let $0 \leq \lambda_\alpha < \infty$, $\alpha \geq 0$. Denote by $G\{\lambda_\alpha\}$ the set of all points $\xi \in \mathbb{R}^n$ such that $|\xi^\alpha| \leq \lambda_\alpha$ for all $\alpha \geq 0$. Then it is easy to see that $G\{\lambda_\alpha\}$ is compact and $G\{h^{|\alpha|}\lambda_\alpha\} = hG\{\lambda_\alpha\}$, $h \geq 0$. Note that $G\{\lambda_\alpha\}$ can be nonconvex. Actually, let $n = 2$, $\lambda_{(i,j)} = 2^{|i-j|}$, $i, j \in \mathbb{Z}_+$. Then $G\{\lambda_{(i,j)}\} = \{(x, y) \in \mathbb{R}^2 : |xy| \leq 1, |x| \leq 2, |y| \leq 2\}$ —the parabola cross.

If there exist $m \geq 1$ and $\beta \geq 0$ such that $\lambda_\beta^m < \lambda_{m\beta}$, then $G\{\lambda_\alpha\}$ does not change when we replace $\lambda_{m\beta}$ by λ_β^m . So, to define $G\{\lambda_\alpha\}$ we can always assume that the sequence $\{\lambda_\alpha\}$ is right, i.e. $\lambda_\alpha \geq \lambda_{m\alpha}^{1/m}$ for all $\alpha \geq 0$ and $m \geq 1$. Using Theorem 1, we can prove the following

Theorem 3. Let $0 < p \leq \infty$, $f(x) \in L_p(\mathbb{R}^n)$ and $\{\lambda_\alpha\}$ be right. Then $\text{sp}(f) \subset G\{\lambda_\alpha\}$ if and only if

$$(37) \quad \overline{\lim}_{|\alpha| \rightarrow \infty} (||D^\alpha f||_p / \lambda_\alpha)^{1/|\alpha|} \leq 1.$$

Proof. Let $\text{sp}(f) \subset G\{\lambda_\alpha\}$. Then

$$\sup_{\text{sp}(f)} |\xi^\alpha| \leq \lambda_\alpha, \quad \alpha \geq 0.$$

Therefore, since Theorem 1, we get (37).

Conversely, if (37) holds, then given Theorem 1 we have

$$\overline{\lim}_{|\alpha| \rightarrow \infty} (\sup_{\text{sp}(f)} |\xi^\alpha| / \lambda_\alpha)^{1/|\alpha|} \leq 1.$$

Therefore, for any $\epsilon > 0$ there exists a number $N < \infty$ such that

$$\sup_{\text{sp}(f)} |\xi^\alpha| \leq (1 + \epsilon)^{|\alpha|} \lambda_\alpha, \quad |\alpha| \geq N.$$

On the other hand, the sequence $\{\lambda_\alpha\}$ is right; therefore,

$$\sup_{\text{sp}(f)} |\xi^\alpha| \leq (1 + \epsilon)^{|\alpha|} \lambda_\alpha$$

for all $\alpha \geq 0$. Hence,

$$\text{sp}(f) \subset (1 + \epsilon)G\{\lambda_\alpha\}.$$

Letting $\epsilon \rightarrow 0$, we get $\text{sp}(f) \subset G\{\lambda_\alpha\}$. The proof is complete.

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